Strategies for Advancing Children’s Mathematical Thinking

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Ms. Smith, a first-grade teacher, asked her students how they could add thirty cents and twenty-seven cents. The students suggested several different solution methods. One student said that they could use the number grid to count. Ms. Smith asked where they should start, and the student said, “At thirty.” Together with the student, Ms. Smith counted twenty-seven more squares on the grid. After finishing, Ms. Smith asked whether the students knew a shortcut. Another student said to start on 27 and count by tens to end at 57.

In this classroom example, the teacher encouraged the students to talk about how they might solve an addition problem, helped one student execute his method, and challenged students to consider alternative methods. This kind of interaction with students stands in sharp contrast with conventional mathematics teaching in which a teacher might ask a closed question, such as “If you have thirty cents and twenty-seven cents, how much do you have altogether?” Here, Ms. Smith engaged the children in mathematical thinking and generated mathematical discussion in the classroom. Her teaching exemplifies instructional ideas presented in the NCTM’s Standards documents (1989, 1991, 2000).

What can we learn from this kind of teaching? How can teachers foster problem-solving skills in children? How can teachers advance children’s thinking while students are engaged in mathematical inquiry? These questions have no easy answers. Fortunately, humans are naturally adept at learning from examples. By studying examples of effective instruction, we can begin to define instructional methods that have proved successful for other classroom teachers (Schifter 1996). To successfully apply lessons from these models, we must answer two principal questions: (1) What are the characteristics of effective teaching? (2) What general principles of instruction help children make sense of mathematics?

In this article, the practice of one teacher who masterfully engages students in mathematics learning serves as an example from which we can draw generalizations. The general principles of this kind of instruction are organized into a pedagogical framework that can guide other teachers as they move toward providing instruction that focuses on children’s thinking. This “Advancing Children’s Thinking” (ACT) framework can help teachers design and implement instruction to make mathematics personally meaningful for children. The ACT framework also establishes a structure for the often-complex interactions that occur when teachers and students grapple with real mathematics problems.
support children’s conceptual understanding, and (3) her skill at extending children’s mathematical thinking. These related teaching components and the supportive classroom climate in which they were used form the basis of the ACT framework (see fig. 1). The following sections describe each of the framework’s components and the underlying learning environment by articulating some of the effective teaching strategies used by Ms. Smith (see Fraivillig, Murphy, and Fuson [1999] for a fuller description of this study).

**Eliciting children’s solution methods**

An important step in advancing children’s thinking is to challenge children to describe and analyze their solution methods. The following paragraphs illustrate instructional strategies that Ms. Smith used to elicit children’s solution methods (see fig. 2).

*Elicit many solution methods for one problem.*

Rather than focus on a single answer to a mathematics problem, Ms. Smith attempted to foster discussion about how students solved a problem. She asked such questions as “Who did this problem another way?” “Did anyone do something else, that Allan did not do?” and “Can you use anything else besides your fingers and a number line to solve this problem?” By asking these types of questions, Ms. Smith encouraged children to share their ideas. Moreover, children in this classroom readily discovered that many approaches are available to solve problems.

*Wait for, and listen to, students’ descriptions of solution methods.* A sense of calm and patience is needed to encourage children to express their ideas. Providing sufficient wait time and listening to ideas let children know that thoughtful explanations are more valuable than quick answers.

*Encourage elaboration.* Often, children need prompting to explain their thinking more completely. Even though Smith may have understood a child’s response, she encouraged clarifications for the benefit of the entire class. On occasion, she assisted students in articulating their methods.

*Use students’ explanations for the lesson’s content.* Students’ articulated ideas can furnish the content of class discussions. Ms. Smith treated the children’s explanations of their solution methods and their mathematical thinking as the content of the lessons themselves. To support this goal, she usually listed the students’ methods on the board to help the class remember and refer to these methods in subsequent discussions.

*Convey an attitude of acceptance toward students’ errors and efforts.* Ms. Smith accepted and often highlighted children’s errors to turn them into “teachable moments.” She explicitly told her students, “Don’t worry about the answer yet,” giving them time to explore and discuss various problem-solving strategies before evaluating the answer. The students in Smith’s class realized that they would not necessarily be judged on the basis of their initial responses and enthusiastically contributed their thoughts. The students’ eagerness to participate may be a result of Smith’s accepting attitude.

*Promote collaborative problem solving.* Smith and her students exhibited a mutual respect for one another and worked as a team to solve problems. An atmosphere of intellectual excitement and team spirit pervaded the classroom when everyone was working on a problem. The students were eager to share their thinking.

*Decide which students need opportunities to report.* The art of facilitating a discussion requires a teacher to decide which students need opportunities to share their ideas. Sometimes this decision is based on prompting reports from a variety of students. Smith prompted many children to participate in the classroom conversation, but more important, she selected students who were able to contribute different solution methods to the discussion.

**Supporting children’s conceptual understanding**

What should a teacher do after eliciting student’s
ideas? One possibility is to assist children in carrying out the solution methods that seem to mesh with their current cognitive abilities, or zones of proximal development (Vygotsky 1978). The supporting component of the ACT framework describes the instructional techniques used to support children’s fragile understanding of their own solution methods, as well as to help them understand the ideas offered by peers. Examples of Smith’s support of children’s conceptual understanding during mathematics instruction are elaborated in the following paragraphs (see fig. 3).

**Remind students of conceptually similar problem situations.** To “jump-start” their thinking, children may need to be reminded that one problem is like another one that they have solved previously.

**Review background knowledge.** Reviewing necessary background knowledge with students is another effective support strategy. For example, Smith reviewed coin values for a student who was having trouble counting money.

**Lead students through instant replays.** Teachers can support the understanding of all children in the class by revisiting one child’s elicited solution method in a slow and step-by-step fashion. This strategy is very different from that of a teacher who offers his or her own solution method as the only sanctioned method.

**Write symbolic representations of each solution method on the board.** Writing the symbols for the children’s solution methods on the board helps children link the verbal descriptions of their thinking with the written mathematical marks. Smith noted another benefit of this strategy: “Recording on the board assists students in following the procedure. Some students must see the numbers. Constant review of this helps them write the digits.”

**Encourage students to request assistance.** Ms. Smith expected children to request additional help when necessary. She attached no stigma to the requirement for extra help; on the contrary, students who requested assistance received extra time and attention from the teacher. This acceptance and expectation is an important aspect of a teacher’s support of students’ learning.

**Extending children’s mathematical thinking**

Eliciting and supporting children’s thinking alone do not challenge children’s mathematical understanding. The extending component of the ACT framework describes teaching strategies that challenge children’s thinking. In this regard, Smith’s teaching was truly exceptional. The children in her classroom experimented with alternative solution methods, analyzed and compared solution methods, and generalized ideas across mathematical concepts. Some of the teaching strategies that she used to encourage this kind of learning are highlighted in the following paragraphs (see fig. 4).

**Maintain high standards and expectations for all students.** Ms. Smith asked all students to...
attempt to solve difficult problems. Students at all levels engaged in problem solving, although the complexity of their solution methods varied and they received different degrees of scaffolding from the teacher. All students, however, contributed to the classroom mathematics community.

Encourage students to draw generalizations. Smith challenged students to move beyond their initial problem-solving efforts and to generalize across mathematical concepts by modeling mathematical thinking. On one occasion, the teacher concluded a class discussion of the different number sentences that could be generated using the numbers 6, 2, and 8. As Smith began to distribute materials for a different activity, a student eagerly exclaimed, “It doesn’t matter which way you put the numbers [6 and 2] together. It will always make the same answer as long as you use the same numbers.” The teacher probed, “Is that true for addition and subtraction?” The first grader immediately began scribbling other number sentences on her paper, motivated by the teacher’s challenge. Smith did not lose an opportunity to extend a student’s thinking, even in the midst of performing management routines. She understood and accepted this student’s expression of the commutative property, and she immediately led the student to extend the concept to test its generality.

List all solution methods on the board to promote reflection. During class discussions, Smith listed the different solution methods offered by the students. This strategy encourages reflection, chronicles the discussion for reference when a student’s concentration wanders, and reinforces the classroom norm of valuing multiple ways of solving a problem.

Push individual students to try alternative solution methods. Smith challenged individual students to try solution methods that differed from their initial attempts. This strategy worked especially well for students who arrived at solution methods with ease. This type of challenge promotes flexibility in students’ mathematical thinking. It also gave Smith a technique for managing the range of student achievement levels in her classroom.

Promote the use of more efficient solution methods. Even when a student was able to explain a solution method, Smith often asked whether the student could find a shorter way. She extended children’s thinking beyond their first attempts. Smith’s students began to understand that although many solution methods are valid, some methods are more efficient than others. Of course, this strategy requires the teacher to be sensitive to children’s current and potential understanding so that students are not pushed beyond what they can do even with appropriate scaffolding.

Cultivate a love of challenge. The feeling of excitement in Smith’s mathematics classroom was infectious. Smith cultivated this attitude by modeling her enthusiasm. She would respond to student-generated problems with such comments as “A challenge! I love it” or “Don’t tell me. I want to figure it out myself.” Smith encouraged students to challenge her and to challenge one another. The students in this class loved to pose difficult questions and to tackle complex problems.
Intersections between and among Components

Although eliciting, supporting, and extending describe elements of effective instruction, the art of teaching is much too complex to be captured by these three components. Classroom discussions and activities are interrelated and serve many functions. In fact, each classroom incident may be described by more than one of these components. For example, eliciting a response from one student may extend the thinking of other students. Smith demonstrated this interaction by following up students’ explanations with such questions as “Does this rule apply in all situations?” or “How did you know that?” She also elicited and extended children’s thinking at the same time by highlighting and discussing children’s errors. For example, Smith pushed one student to examine his explanation of an incorrect solution method for determining the difference between 28 and 52. This strategy resulted in a productive whole-class mathematical discussion and one student’s identification of a rule for using diagonals on the number grid. Using errors as learning opportunities was a hallmark of Smith’s teaching.

Providing Learning Opportunities in a Safe Environment

The teaching practices described here could not be effective unless they occurred in a safe environment. Smith’s efforts to elicit, support, and extend children’s thinking were successful because she accepted children’s ideas in a rational, nonthreatening manner and incorporated students’ contributions into intellectual discussions. She established this environment by modeling respect for each child’s thinking. She insisted that students listen to and respect other children’s comments and questions. Smith explained this aspect of her philosophy:

Students may not laugh at another student who is asking a question. That is the only time a student may be put out of the room. Without this rule, students would stop asking questions, which stops learning. Then students feel safe to explore, to do things. Then they are more available to learn and discuss.

Smith positively reinforced her students for expressing their ideas and emphasized each student’s individual contributions to the classroom mathematics community.

Conclusion

Although no road map exists for reforming mathematics teaching, the ACT framework offers teachers a compass to guide their instruction. The framework highlights the goal of helping children create their own meaningful understanding of mathematical concepts. The eliciting component reminds teachers to consider how they might get children’s thinking out in the open for discussion and build instruction on that thinking. The supporting component describes instructional strategies for assisting children at their current levels of understanding. The extending component prompts teachers to challenge children’s thinking regardless of the student’s initial efforts. Of course, the complex art of teaching for meaningful mathematical learning is highly personal; each teacher must incorporate these guidelines into his or her own instructional style. The ACT framework can serve teachers who wish to create classrooms in which all children’s mathematical thinking is elicited, supported, and extended.

References


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