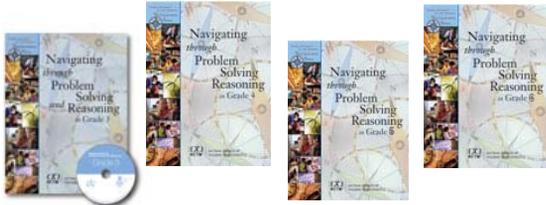


## Navigating through Problem Solving and Reasoning

Seven grade level books (PreK- 6 grades) that present investigations designed to develop students' reasoning and problem solving strategies.



by Michael Battista, Sally Mayberry, Denisse R. Thompson,  
Karol L. Yeatts, and Judith S. Zawajewski

## NCTM Standards

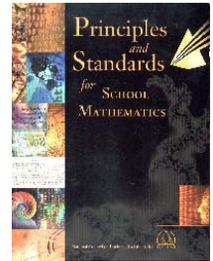
- Five investigations are included in each grade level book.
- Each investigation is situated in a different one of the five content strands.

### Content Standards

- Number and Operations
- Algebra
- Geometry
- Measurement
- Data Analysis and Probability

### Process Standards

- Problem Solving
- Communication
- Connections
- Representation
- Reasoning & Proof



## NCTM Process Standards

### Problem Solving

*Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Build new mathematical knowledge through problem solving
- Solve problems that arise in mathematics and in other contexts
- Apply and adapt a variety of appropriate strategies to solve problems
- Monitor and reflect on the process of mathematical problem solving

### Reasoning and Proof

*Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Recognize reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof

## NCTM Process Standards

### Communication

*Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Organize and consolidate their mathematical thinking through communication
- Communicate their mathematical thinking coherently and clearly to peers, teachers, and others
- Analyze and evaluate the mathematical thinking and strategies of others;
- Use the language of mathematics to express mathematical ideas precisely.

### Connections

*Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Recognize and use connections among mathematical ideas
- Understand how mathematical ideas interconnect and build on one another to produce a coherent whole
- Recognize and apply mathematics in contexts outside of mathematics

### Representation

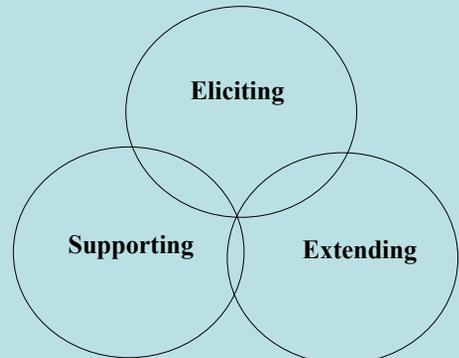
*Instructional programs from prekindergarten through grade 12 should enable all students to—*

- Create and use representations to organize, record, and communicate mathematical ideas
- Select, apply, and translate among mathematical representations to solve problems
- Use representations to model and interpret physical, social, and mathematical phenomena

## Strategies for Advancing Children's Mathematical Thinking

- Judith Fravillig (2001) has identified various teacher behaviors that are critical to helping students think deeply about mathematical ideas and share their thinking with others.
- These behaviors are summarized in three broad categories: eliciting students' thinking, supporting students' thinking, and extending students' thinking.
- **Fravillig, Judith. "Strategies for Advancing Children's Mathematical Thinking." *Teaching Children Mathematics*, April 2001 pages 454-459**

## Advancing Children's Thinking Framework



## Eliciting students' thinking

- Elicit many solution methods for one problem.
- Wait for, and listen to, students' descriptions of solution methods.
- Encourage elaboration and student discussion.
- Use students' explanations as a basis for the lesson's content.
- Convey an attitude of acceptance toward students' errors and efforts.
- Promote collaborative problem solving.

Supporting

Extending

## Supporting students' thinking

Eliciting

Extending

- Remind students of conceptually similar problem situations.
- Provide background knowledge.
- Lead students through instant replays. (Revisit student solutions.)
- Write symbolic representations of solutions when appropriate.

## Extending students' thinking

Eliciting

Supporting

- Maintain high standards and expectations for all students.
- Encourage students to draw generalizations.
- List all solution methods on the board to promote reflection.
- Push individual students to try alternative solution methods.
- Promote use of more-efficient solution methods.

## Using Questioning to Stimulate Mathematical Thinking

### • Article by Jenni Way

– [http://nrich.maths.org/public/viewer.php?obj\\_id=2473&part=index&refpage=monthindex.php](http://nrich.maths.org/public/viewer.php?obj_id=2473&part=index&refpage=monthindex.php)

- These questions can be used by the teacher to guide the children through investigations while stimulating their mathematical thinking and gathering information about their knowledge and strategies.

## Starter questions

- These take the form of open ended questions which focus the children's thinking in a general direction and give them a starting point.



### Examples:

- How could you sort these.....?
- How many ways can you find to ..... ?
- What happens when we ..... ?
- What can be made from....?
- How many different ..... can be found?

## Questions to stimulate mathematical thinking

- These questions assist children to focus on particular strategies and help them to see patterns and relationships.
- This aids the formation of a strong conceptual network.
- These questions can also serve as a prompt when children become stuck.

### Examples:

- What is the same?
- What is different?
- Can you group these ..... in some way?
- Can you see a pattern?
- How can this pattern help you find an answer?
- What do think comes next? Why?
- Is there a way to record what you've found that might help us see more patterns?
- What would happen if....?



## Assessment questions

- Questions such as these ask children to explain what they are doing or how they arrived at a solution.
- They allow the teacher to see how the children are thinking, what they understand and what level they are operating at.
- Obviously they are best asked after the children have had time to make progress with the problem, to record some findings and perhaps achieved at least one solution.

Examples:

- What have you discovered?
- How did you find that out?
- Why do you think that?
- What made you decide to do it that way?



## Final discussion questions

- These questions draw together the efforts of the class and prompt sharing and comparison of strategies and solutions.
- This is a vital phase in the mathematical thinking processes.
- It provides further opportunity for reflection and realization of mathematical ideas and relationships.
- It encourages children to evaluate their work.

Examples:

- Who has the same answer/ pattern/ grouping as this?
- Who has a different solution?
- Are everybody's results the same?
- Why/why not?
- Have we found all the possibilities?
- How do we know?
- Have you thought of another way this could be done?
- Do you think we have found the best solution?

## Practicing Effective Questioning

Maximize...	Minimize...
...asking questions that begin with words like "What if," "Explain," "Analyze," "Create," and "Compare and contrast," etc.	...asking questions that have a "yes" or "no" response and questions that require merely direct recall of definitions etc.
...the amount of time you wait after you pose a question, i.e. wait-time, in order to allow students to process the question in their minds.	...calling on students directly after you pose a question and calling on a student before you even ask the question.
...asking students to elaborate on their answers and asking students "why."	...telling a student their answer is wrong and not asking them to think of why it is wrong.
...opportunities for students to pose questions amongst themselves.	...straight lecture without student interaction.
...providing opportunities that challenge students' original conceptual understandings.	...providing opportunities that do not encourage creative and critical thinking.
...encouraging students to work through their decision making process, even if it bring frustration and makes them leave their comfort zone of learning.	...giving students direct answers to their questions without allowing them to think through the decision making process.

[http://www.ndt-ed.org/TeachingResources/ClassroomTips/Effective\\_Questioning.htm](http://www.ndt-ed.org/TeachingResources/ClassroomTips/Effective_Questioning.htm)

## Developing Mathematical Thinking with Effective Questions

- PBS TeacherLine 
  - <http://teacherline.pbs.org/teacherline/resources/math.cfm>
  - [http://teacherline.pbs.org/teacherline/resources/questionsheet\\_vma.pdf](http://teacherline.pbs.org/teacherline/resources/questionsheet_vma.pdf)



## Effective Use of Questioning

**Questioning can.....**

- arouse curiosity
- stimulate interest in the topic
- clarify concepts
- emphasize key points
- enhance problem solving ability
- encourage students to think at higher cognitive levels
- motivate student to search for new information
- ascertain students' knowledge level to aid in modifying instruction



## Problem Solving

- When students use problem-solving skills, ***"The role of the student changes from a passive recipient of information to a participant in the creation of understanding. The problem should captivate students' attention, be meaningful, and allow a wide range of individual responses."***
  - Rossman, Alan D. (1993). *Managing Hands-On Inquiry*. Science and Children, September, page 35.

## NCTM Navigation Series

- All investigations have the same format.
- For each investigation, the focus is identified and the activities are summarized.
- The goals, content strand, prerequisite knowledge and materials are specified.
- Blackline masters are included for many of the activities.
- Each book has a CD ROM that contains the blackline masters, resources for professional development and applets for students to explore.

## NCTM Problem Solving and Reasoning Navigations Series

- Seven grade level (PreK-6<sup>th</sup> grades) books that present investigations designed to develop students' reasoning methods and problem solving strategies.
- Five investigations are included in each grade level book.
- Each investigation is situated in a different one of the five content strands.

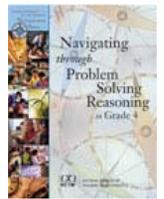
## 3rd Grade Investigations

- **Walking into Place Value**
  - Reasoning about Number Relationships
- **And We All Go Marching**
  - Reasoning about Algebraic Relationships
- **Cut It Apart, Put It Together**
  - Reasoning about Geometric Thinking
- **How Many Are Too Many**
  - Reasoning about Measurement Relationships
- **Grant Avenue Elementary School Reading Certificates**
  - Reasoning about Data Relationships



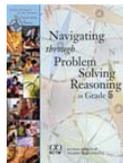
## 4th Grade Investigations

- **Discovering Primes**
  - Reasoning about Number Relationships
- **Movie Money Matters**
  - Reasoning about Algebraic Relationships
- **Making and Investigating Puzzles**
  - Reasoning about Geometric Thinking
- **Fascinating Measures**
  - Reasoning about Measurement Relationships
- **Growing Giant Sequoias**
  - Reasoning about Data Relationships



## 5th Grade Investigations

- **Pin Numbers and Secret Codes**
  - Reasoning about Number Relationships
- **Carina's Pet Shop**
  - Reasoning about Algebraic Relationships
- **Making and Breaking Solids**
  - Reasoning about Geometric Thinking
- **Explain that Measure**
  - Reasoning about Measurement Relationships
- **Collecting Data**
  - Reasoning about Data Relationships



## Organization of Lessons



## 3rd Grade Investigation And We All Go Marching

- Focus
  - Reasoning about Algebraic Relationships
- Summary
  - Students will begin to understand how a change in one quantity produces a change in a second quantity.
- Materials
  - copies of the Blackline masters “Centimeter Grid Paper” and “Parade Formation Chart.”
  - Centimeter cubes

## And We All Go Marching

### Goals

- Investigate the number 36 by representing and analyzing an arrangement of thirty ~~x~~ objects
- Model and explain how a change in one quantity produces a change in a second quantity.

## And We All Go Marching

### Mathematical Content

This investigation supports the following Process and Algebra Standards and expectations for grades 3–5 (NCTM 2000, pp. 394, 402):

#### Algebra

- Understand patterns, relations, and functions
  - Describe, extend, and make generalizations about geometric and numeric patterns
- Represent and analyze mathematical situations and structures using algebraic symbols
  - Identify such properties as commutativity, associativity, and distributivity and use them to compute with whole numbers
- Use mathematical models to represent and understand quantitative relationships
  - Model problem situations with objects and use representations such as graphs, tables, and equations to draw conclusions

## And We All Go Marching

### Prior Knowledge

- Skip-counting by twos, threes, fours, fives, and tens
- Fluency with basic number combinations for addition and subtraction
- Constructing tables, charts, and graphs

## And We All Go Marching

### Materials

- Linking cubes, centimeter cubes, colored tiles, or counters for students to use to model all the possible marching arrangements.
- For each student, a copy of the blackline master “[Grid Paper](#)” on which to represent the possible marching arrangements.
- A transparency copy of “[Grid Paper](#)”
- An overhead projector
- Colored pencils, markers or crayons for each student
- A copy of the blackline master “[Parade-Formation Chart](#)” for each student
- For each student, a copy of the blackline master, “[Centimeter Grid Paper](#),” on which to draw the graphs of the relationships
- Paper and pencil for each student
- Chart paper and a marker (optional)

## And We All Go Marching

### Classroom Environment

- The students work individually or in groups on the components of the contest after hearing the opening scenario and discussing the directions for each of the components.



## And We All Go Marching

### Investigation

- Present the following scenario to the students:
- The Super Bowl Committee is sponsoring a contest to select 36 third grade students to march in the pre-game parade.
- The contest has 5 components.
- Entrants must complete each part to compete in the contest.
- If your entry wins you will be one of the 36 students chosen to march in the parade.
- GOOD LUCK!



## And We All Go Marching

- Step 1
  - You must determine all the possible marching arrangements for 36 marchers.
  - Each marching arrangement must have equal numbers of students marching in all the rows.
  - You must show the marching arrangements with a model such as linking cubes or centimeter cubes

## And We All Go Marching

- Step 2
  - Entrants must represent the marching arrangements on the contest Grid Paper.
- Step 3
  - Using the “Parade-Formation Chart” entrants must display each marching arrangement showing the number of rows and the number of marchers in each row.
- Step 4
  - Using the “Grid Paper” entrants must draw a graph on representing the marching arrangements.
- Step 5
  - Entrants must submit a written mathematical explanation of the relationship among the different arrangements of 36 marchers.



## And We All Go Marching

### Assessment

- Your assessment will consist primarily of informal observations of students as they construct their models, complete the chart, and make their graphs.
- Giving the students an opportunity to share their written mathematical explanations will allow you to assess the students’ understanding of how a change in one variable affects another variable.



## And We All Go Marching

### Reflection

- This investigation leads students to discover how a change in one quantity affects a second quantity and helps them make connections among physical, graphical, and verbal descriptions of change.
- Seeing the relationship between variables is essential to understanding change.
- This activity also promotes students’ development of flexibility of flexibility in reasoning and illustrates the commutative property of multiplication.
- Experiences such as those in this investigation offer students an opportunity to represent algebraic ideas in a real-life context and give them experience in using various reasoning skills to solve a complex problem.



## And We All Go Marching

### Connections

- After having carried out this investigation, students should be better equipped to describe change both qualitatively and quantitatively, and they should begin to apply the idea of change as they measure phenomena over time.
- Understanding that many changes are predictable helps lay a foundation for applying mathematics to other subjects.
- As part of a science unit, for example, students might plant seed and record the growth of a plant on a chart and then graph the data.
- Using their observations and records, they might then describe how the rate of growth varies over time.
- In the higher grades, students will continue to develop an understanding of change as they study situations that display constant, increasing, and decreasing rates of change.

## Cut it Apart, Put it Together

- Focus
  - Reasoning about Geometric Relationships
- Summary
  - Students explore how they can decompose shapes to make other shapes.
- Materials
  - Copy of Blackline Masters
    - Polygon Pizzas, One Cut Pizzas and Make the Pizzas
  - Tangrams
  - Scissors, tape and rulers
  - [CD-ROM 2-D-Shape Decomposition Tool](#)

## Cut it Apart, Put it Together

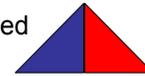
### Task 1: Polygon Pizzas

- Have students use two small right triangles and the square from a set of tangrams to make and reason about the shapes that can be constructed from these pieces.



- Note: Pieces must match along an edge.

Allowed



Not Allowed



## Cut it Apart, Put it Together

### Task 2: One-Cut Pizza

- Have students decompose an 8"x 6" rectangle to make various shapes on dot paper.
1. Make one straight cut to cut the rectangular pizza into two rectangles.
  2. Make one straight cut to cut the rectangular pizza into two triangles.
  3. Make one straight cut to cut the rectangular pizza into two four-sided pieces that are congruent.
  4. Make one straight cut to cut the rectangular pizza into two different four-sided shapes.
  5. Make one straight cut to cut the rectangular pizza into one triangle and a five-sided shape.



## Cut it Apart, Put it Together

### Task 3: Make the Pizzas

- Have students decompose each pizza into two pieces that can be recomposed to make another pizza.
1. Make one straight cut to cut pizza B into pieces that can be used to make pizza A.
  2. Make one straight cut to cut pizza C into pieces that can be used to make pizza A.
  3. Make one straight cut to cut pizza D into pieces that can be used to make pizza A.
  4. Make one straight cut to cut pizza E into pieces that can be used to make pizza A.

## 4th Grade Investigation Discovering Primes

- Focus
  - Reasoning about Number Relationships
- Summary
  - Students explore the concepts of factors and primes as they begin to relate these concepts to real life.
- Materials
  - Hundred's Chart
  - Markers or crayons

## 4th Grade Investigation Discovering Primes

- Provide each student with a copy of the Sieve of Eratosthenes Directions, and the 100's chart.
- Have students predict how many prime numbers exist between 1 and 100.
- Have students follow the directions on the Blackline master as they discover the solution.
- After students have completed their search for prime numbers have the students create a set of Divisibility Rules based on what they have discovered.

## Pin Numbers and Secret Codes

- Focus
  - Reasoning about Number Relationships
- Summary
  - Students learn about PIN numbers using the divisibility rules to promote problem solving and reasoning and enhance the understanding of divisibility.
- Materials
  - [Copy of BLM "Chart for Discovering the Divisibility Rules"](#)
  - Calculators (class set)



## Pin Numbers and Secret Codes

- Jamie has recently installed a new alarm system and needs to create a secret password (PIN number) for the alarm.
- She decides to use the numbers in her birthday: **05-24-76** for her six-digit PIN number.
- It is of the utmost importance to Jamie that no one else be able to decipher her secret code so she decides to rearrange this special 6-digit number in a way that only she can remember it.



## For More Information

- Visit the National Council of Teachers of Mathematics webpage:
  - [www.nctm.org](http://www.nctm.org)
- NCTM Catalog
  - <http://my.nctm.org/eBusiness/ProductCatalog/product.aspx?ID=12719>
  - <http://my.nctm.org/ebusiness/ProductCatalog/product.aspx?ID=12886>